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ZW 1953-25r

A property of logarithmic concave functions, 1-2.

C.G. Lekkerkerker.

Reprinted from:
Proceedings KNAW Series A, 56 (1953), nr. 5,
Indagationes Mathematicae, 15 (1953),
p 505-521.

1953

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 56, No. 5 and Indag. Math., 15, No. 5, 1953

MATHEMATICS

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A PROPERTY OF LOGARITHMIC CONCAVE FUNCTIONS. I

BY

C. G. LEKKERKERKER

(Communicated by Prof. J. G. van der Corput at the meeting of October 31, 1953)

In this paper we use the following definitions.

Definition I. A real function $g(x)$ of a real variable x is said to be *convex* in the interval (a, b) if in this interval $g(x)$ satisfies the condition

(1) $$g(x) \leqq \frac{x_2-x}{x_2-x_1}\, g(x_1) + \frac{x-x_1}{x_2-x_1}\, g(x_2) \ \textit{if}\ x_1 < x < x_2;$$

$g(x)$ is said to be *concave*, if this condition holds with the $\leqq$ sign replaced by the $\geqq$ sign.

Definition II. A positive function $f(x)$ is called *logarithmic convex* (*logarithmic concave*) in the interval (a, b) if $g(x) = \log f(x)$ is convex (concave) in (a, b). If for each admissable set x_1, x, x_2 the relation (1) holds with the $<$ sign ($>$ sign), then $g(x)$ is called *strictly convex* (*strictly concave*) and $f(x)$ is called *strictly logarithmic convex* (*strictly logarithmic concave*).

Some well-known properties may be stated as follows.

(i) If $g(x)$ is convex, then $-g(x)$ is concave, and conversely.

(ii) If $g(x)$ is convex or concave in the interval (a, b), then $g(x)$ is continuous in the interior of (a, b).

(iii) If $f(x)$ and $\varphi(x)$ are logarithmic convex in (a, b), then the sum $f(x)+\varphi(x)$ also is logarithmic convex in (a, b).

(iiii) Let $f(x, t)$ be a positive function of two real variables x, t. Let (a, b) and (c, d) be two intervals, such that for each t in the interval (c, d), $f(x, t)$ is a logarithmic convex function of x in (a, b), and such that

$$F(x) = \int_c^d f(x, t)\, dt$$

exists if x belongs to (a, b). Then the function $F(x)$ is logarithmic convex in (a, b).

Generally spoken (iii) and (iiii) are not true for logarithmic concave functions. For a special class of functions, however, it may be possible to establish the analogues of (iii) and (iiii). The main object of this paper is a proof of the following remarkable result.

Theorem 1. *Let $f(x)$ and $\varphi(x)$ be two functions of a real variable x Suppose that*

1^0. *$f(x)$ and $\varphi(x)$ are positive and steadily decreasing for $x \geqq 0$.*

2^0. *$f(x)$ and $\varphi(x)$ are logarithmic concave in the interval $0 \leqq x < \infty$.*

Then the integral

$$(2) \qquad \int_0^\infty f(t)\varphi(t+x)\,dt$$

exists and represents a function $f_1(x)$, *which likewise is positive, steadily decreasing and logarithmic concave in the interval* $0 \leqq x < \infty$.

Furthermore, if $\log \varphi(x)$ *is not a linear function of* x *in any interval* $a \leqq x < \infty$ $(a > 0)$, *then this function* $f_1(x)$ *is strictly logarithmic concave.*

As a consequence of theorem 1 I prove

Theorem 2. *Let* $f(x)$ *be continuous, positive, steadily decreasing and logarithmic concave in the interval* $0 \leqq x < \infty$. *Suppose that* $\log f(x)$ *is not linear in the interval* $0 \leqq x < \infty$ *and put*

$$(3) \qquad \int_0^\infty f(x)\,dx = \gamma.$$

Then γ *is finite and by the relations*

$$(4) \qquad f_0(x) = f(x)\ ,\ f_{n+1}(x) = \frac{2}{\gamma}\int_0^\infty f_n(t) f_n(t+x)\,dt \quad (n = 0, 1, \ldots)$$

a sequence of functions is defined, for which

$$(5) \qquad 0 < f_0(0) < f_1(0) < f_2(0) < \ldots .$$

We give an application of the last theorem. Let $x_1 = x_1^{(0)}$, $x_2 = x_2^{(0)}$, ... be random variables, independently distributed with common density function $f_0(x)$. We suppose that $f_0(x)$ is symmetric and that for $x \geqq 0$ this function is continuous and steadily decreasing. We next consider the random variables $x_k^{(n)}$, defined inductively by

$$x_k^{(n)} = |x_{2k-1}^{(n-1)}| - |x_{2k}^{(n-1)}| \qquad (k = 1, 2, \ldots;\ n = 1, 2, \ldots).$$

For fixed n the random variables $x_1^{(n)}$, $x_2^{(n)}$, ... are independently distributed with a common density function, which may be denoted by $f_n(x)$. Since the density function $f_n^*(x)$ of $|x_k^{(n)}|$ is given by

$$f_n^*(x) = \begin{cases} 2 f_n(x) & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

we have the formula

$$f_{n+1}(x) = 4\int_0^\infty f_n(t)\, f_n(t+|x|)\,dt \quad (n = 1, 2, \ldots).$$

We ask for a condition, such that the values $f_0(0)$, $f_1(0)$, $f_2(0)$, ... form a monotoneously increasing sequence [1]). In virtue of the symmetry of $f_0(x)$ we have

$$\int_0^\infty f_0(x)\,dx = \tfrac{1}{2}\int_{-\infty}^\infty f_0(x)\,dx = \tfrac{1}{2}.$$

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[1]) This question was raised by Mr J. DE BOER, collaborator at the Statistical Department of the Mathematical Centre at Amsterdam. After a discussion with Prof. V. D. CORPUT and Prof. VAN WIJNGAARDEN I was led to the assertions of theorems 1 and 2.

Now an answer to the question is afforded immediately by theorem 2. For taking $\gamma = \frac{1}{2}$, we see that the sequence of values $f_n(0)$ is monotoneously increasing indeed, if $f(x)$ satisfies the additional condition of being logarithmic concave in the interval $0 \leqq x < \infty$, whereas $\log f(x)$ is not linear in any interval $a \leqq x < \infty$ $(a > 0)$.

Proof of theorem 1.

1. *The integral* (2) *exists for* $x \geqq 0$, *and, as a function of* x, *is positive and steadily decreasing*

By (ii) and 2^0 we see that $f(x)$ and $\varphi(x)$ are continuous for $x > 0$. Clearly it is no loss of generality to suppose that these functions are continuous also at the point $x = 0$ and there assume the value

$$f(0) = \varphi(0) = 1. \tag{6}$$

Put $g(x) = -\log f(x)$, $g(1) = \alpha$. Then $g(x)$ is steadily increasing and convex in the interval $0 \leqq x < \infty$. In particular we have $\alpha > g(0) = 0$. Applying (1) with $x_1 = 0$, $x = 1$, $x_1 > 1$ we find

$$\alpha \leqq \frac{x-1}{x}\, g(0) + \frac{1}{x}\, g(x) = \frac{1}{x}\, g(x) \ \ for\ \ x > 1,$$

hence

$$f(x) = e^{-g(x)} \leqq e^{-\alpha x} \ \ for\ \ x \geqq 1.$$

Similarly there exists a positive number β, such that

$$\varphi(x) \leqq e^{-\beta x} \ \ for\ \ x \geqq 1.$$

Consequently the integral (2) exists [2]). Let it represent the function $f_1(x)$. Evidently $f_1(x)$ is positive. For $0 \leqq x_1 < x_2$ we have

$$\varphi(t + x_1) > \varphi(t + x_2) \ \ for\ all\ \ t \geqq 0,$$

hence

$$\int_0^\infty f(t)\, \varphi(t + x_1)\, dt > \int_0^\infty f(t)\, \varphi(t + x_2)\, dt,$$

i.e. $f_1(x_1) > f_1(x_2)$. Hence $f_1(x)$ is steadily decreasing.

2. *Differentiability of* $f_1(x)$ *in a special case*

We now consider functions $f(x)$, $\varphi(x)$ of a more special kind. In fact we shall suppose, in this and the next three sections, that $f(x)$ and $\varphi(x)$ instead of 1^0, 2^0 fulfill the more restrictive conditions

3^0. *$f(x)$ and $\varphi(x)$ are continuous for $x \geqq 0$, whereas*

$$f(0) = \varphi(0) = 1 \tag{6}$$

4^0. *there exist a positive number σ and four sequences of positive numbers α_n, β_n, c_n, d_n $(n = 1, 2, \ldots)$, such that*

$$0 < \alpha_1 \leqq \alpha_2 \leqq \ldots \ ; \ 0 < \beta_1 \leqq \beta_2 \leqq \ldots \tag{7}$$

$$\left.\begin{aligned} f(x) &= c_n\, e^{-\alpha_n x} \\ \varphi(x) &= d_n\, e^{-\beta_n x} \end{aligned}\right\} \ if\ \ (n-1)\, \sigma \leqq x < n\sigma \quad (n = 1, 2, \ldots). \tag{8}$$

[2]) For the same reason the number γ occurring in theorem 2 is finite.

We note that according to the continuity of $f(x)$ and $\varphi(x)$ the numbers c_n, d_n satisfy the relations

$$\left.\begin{aligned} f(n\sigma) &= c_n\, e^{-\alpha_n\cdot n\sigma} = c_{n+1}\, e^{-\alpha_{n+1}\cdot n\sigma} \\ \varphi(n\sigma) &= d_n\, e^{-\beta_n\cdot n\sigma} = d_{n+1}\, e^{-\beta_{n+1}\cdot n\sigma} \end{aligned}\right\} \qquad (n = 1, 2, \ldots), \tag{9}$$

whereas

$$c_1 = f(0) = 1 \;,\; d_1 = \varphi(0) = 1. \tag{10}$$

In this section we shall prove that, *if* $f(x)$ *and* $\varphi(x)$ *fulfill the conditions* 3^0 *and* 4^0, *the function*

$$f_1(x) = \int_0^\infty f(t)\, \varphi(t+x)\, dt \tag{11}$$

s continuously differentiable for $x \geqq 0$.

Let $x_0 > 0$ be arbitrary and let ε be a positive number. Put

$x_0 = g\sigma + \xi_0$, where $0 \leqq \xi_0 < \sigma$ and g is a non-negative integer

$t_n = n\sigma - \xi_0$ $(n = 1, 2, \ldots)$,

so that $t_1 > 0$.

Let δ and h be any real numbers with

$$0 < \delta < \tfrac{1}{2}\sigma \;,\; \delta < t_1 \;,\; h \neq 0 \;,\; x_0 + h \geqq 0.$$

Then we may write

$$\frac{f_1(x_0+h) - f_1(x_0)}{h} = \int_0^\infty f(t) \cdot \frac{\varphi(t+x_0+h) - \varphi(t+x_0)}{h}\, dt$$

$$= \int_0^{t_1-\delta} + \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n+\delta} + \sum_{n=1}^{\infty} \int_{t_n+\delta}^{t_{n+1}-\delta} .$$

In virtue of the conditions 3^0 and 4^0 we clearly have

$$\left| \frac{\varphi(t+h_1) - \varphi(t)}{h_1} \right| \leqq \max_{n=1,2,\ldots} \beta_n\, e^{-\beta_n\cdot(n-1)\sigma} = B, \text{ say,}$$

for all real t and h_1 with $t \geqq 0$, $h_1 \neq 0$, $t + h_1 \geqq 0$. Hence

$$\int_{t_n-\delta}^{t_n+\delta} f(t) \cdot \left| \frac{\varphi(t+x_0+h) - \varphi(t+x_0)}{h} \right| dt < 2\, B\, \delta\, f(t_n - \delta).$$

Now fix δ, such that

$$2\, B\, \delta \cdot \{ f(0) + \sum_{n=0}^{\infty} f(n\sigma) \} < \varepsilon.$$

Then we get

$$\left| \int_0^{t_1-\delta} f(t) \cdot \frac{\varphi(t+x_0+h) - \varphi(t+x_0)}{h}\, dt + \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n+\delta} f(t) \cdot \frac{\varphi(t+x_0+h) - \varphi(t+x_0)}{h}\, dt \right| < \varepsilon.$$

The function $\varphi(t)$ has a continuous derivative, except possibly at the points $0, \sigma, 2\sigma, \ldots$, whereas

$$|\varphi'(t)| \leqq B \;\; (t \neq 0, \sigma, 2\sigma, \ldots).$$

Hence the function $f(t)$. $|\varphi'(t+x_0)|$ can be integrated over the interval $0 \leqq t < \infty$. In particular we find the estimate

$$\left| \int_{t_n-\delta}^{t_n+\delta} f(t)\cdot\varphi'(t+x_0)\,dt \right| < 2B\delta f(t_n-\delta)$$

and a similar estimate for $\int_0^{t_1-\delta}$. This implies

$$\left| \int_0^{t_1-\delta} f(t)\cdot\varphi'(t+x_0)\,dt + \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n+\delta} f(t)\cdot\varphi'(t+x_0)\,dt \right| < \varepsilon.$$

According to the choice of t_n the derivative $\varphi'(t+x_0)$ certainly exists if $t \neq t_n$. We can find a positive number $\eta < \delta$, x_0, such that

$$|\varphi'(t+x) - \varphi'(t+x_0)| < \varepsilon,$$

if $x_0 - \eta < x < x_0 + \eta$ and if t belongs to the interval $(0, t_1 - \delta)$ or to one of the intervals $(t_n + \delta, t_{n+1} - \delta)$ $(n = 1, 2, \ldots)$. Henceforth

$$\left| \left(\int_0^{t_1-\delta} + \sum_{n=1}^{\infty} \int_{t_n+\delta}^{t_{n+1}-\delta} \right) f(t) \cdot \left\{ \frac{\varphi(t+x_0+h)-\varphi(t+x_0)}{h} - \varphi'(t+x_0) \right\} dt \right|$$

$$< \varepsilon \left(\int_0^{t_1-\delta} f(t)\,dt + \sum_{n=1}^{\infty} \int_{t_n+\delta}^{t_{n+1}-\delta} f(t)\,dt \right) < \varepsilon \int_0^{\infty} f(t)\,dt \;\; if \;\; 0 < |h| < \eta.$$

Combining the results we obtain

$$\left| \frac{f_1(x_0+h)-f_1(x_0)}{h} - \int_0^{\infty} f(t)\cdot\varphi'(t+x_0)\,dt \right|$$

$$= \left| \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n+\delta} f(t) \cdot \frac{\varphi(t+x_0+h)-\varphi(t+x_0)}{h}\,dt - \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n+\delta} f(t)\cdot\varphi'(t+x_0)\,dt + \right.$$

$$\left. + \left(\int_0^{t_1-\delta} + \sum_{n=1}^{\infty} \int_{t_n+\delta}^{t_{n+1}-\delta} \right) f(t) \cdot \left\{ \frac{\varphi(t+x_0+h)-\varphi(t+x_0)}{h} - \varphi'(t+x_0) \right\} dt \right|$$

$$< \varepsilon \left(2 + \int_0^{\infty} f(t)\,dt\right) \;\; if \;\; 0 < |h| < \eta;$$

$$\left| \int_0^{\infty} f(t)\cdot\varphi'(t+x)\,dt - \int_0^{\infty} f(t)\cdot\varphi'(t+x_0)\,dt \right|$$

$$< \varepsilon \left(2 + \int_0^{\infty} f(t)\,dt\right) \;\; if \;\; |x - x_0| < \eta,\ x \geqq 0.$$

We conclude that the function $f_1(x)$, given by (11), is continuously differentiable for $x > 0$, whereas we have

$$f_1'(x) = \int_0^{\infty} f(t)\,\varphi'(t+x)\,dt. \tag{12}$$

By a small modification of the foregoing argument we see that these conclusions even hold for $x \geqq 0$.

3. *An expression for* $f_1(x)$

We suppose $0 < x < \sigma$ and express $f_1(x)$ in terms of σ, α_n, β_n, x. It will appear that $f_1(x)$ is indefinitely differentiable for $0 < x < \sigma$.

Dividing in (11) the range of integration into the intervals $((n-1)\sigma, n\sigma-x)$, $(n\sigma-x, n\sigma)$ $(n=1, 2, \ldots)$ we find from (8)

$$\begin{aligned} f_1(x) &= \sum_{n=1}^{\infty} \int_{(n-1)\sigma}^{n\sigma-x} c_n\, d_n\, e^{-\alpha_n t-\beta_n(t+x)}\, dt + \sum_{n=1}^{\infty} \int_{n\sigma-x}^{n\sigma} c_n\, d_{n+1}\, e^{-\alpha_n t-\beta_{n+1}(t+x)}\, dt \\ &= \sum_{n=1}^{\infty} \frac{c_n d_n}{\alpha_n+\beta_n} \{ e^{-\alpha_n(n-1)\sigma-\beta_n(n-1)\sigma-\beta_n x} - e^{-\alpha_n n\sigma-\beta_n n\sigma+\alpha_n x} \} + \\ &+ \sum_{n=1}^{\infty} \frac{c_n d_{n+1}}{\alpha_n+\beta_{n+1}} \{ e^{-\alpha_n n\sigma-\beta_{n+1} n\sigma+\alpha_n x} - e^{-\alpha_n n\sigma-\beta_{n+1} n\sigma-\beta_{n+1} x} \}. \end{aligned}$$

Using (9) and (10) we can express c_n, d_n in terms of σ, α_n, β_n. We find

$$c_n\, e^{-\alpha_n n\sigma} = e^{-\alpha_n\sigma}\cdot c_n\, e^{-\alpha_n(n-1)\sigma} = e^{-\alpha_n\sigma}\cdot c_{n-1}\, e^{-\alpha_{n-1}(n-1)\sigma} \qquad (n = 2, 3, \ldots),$$

hence

$$c_n\, e^{-\alpha_n n\sigma} = e^{-(\alpha_n+\alpha_{n-1}+\ldots\alpha_2)\sigma}\cdot c_1\, e^{-\alpha_1\sigma} = e^{-(\alpha_1+\alpha_2+\ldots+\alpha_n)\sigma} \qquad (n = 1, 2, \ldots).$$

Similarly we find

$$\left.\begin{aligned} d_n\, e^{-\beta_n n\sigma} \\ d_{n+1}\, e^{-\beta_{n+1} n\sigma} \end{aligned}\right\} = e^{-(\beta_1+\beta_2+\ldots+\beta_n)\sigma} \qquad (n = 1, 2, \ldots).$$

In order to obtain a neat expression for $f_1(x)$ we introduce

$$\left.\begin{aligned} &(13) \qquad U_n(x) = e^{-(\alpha_1+\ldots+\alpha_{n-1})\sigma-(\beta_1+\ldots+\beta_{n-1})\sigma-\beta_n x} \\ &(14) \qquad V_n(x) = e^{-(\alpha_1+\ldots+\alpha_n)\sigma-(\beta_1+\ldots+\beta_n)\sigma+\alpha_n x} \end{aligned}\right\} (n = 1, 2, \ldots)$$

$$(15) \qquad A_1 = -\frac{1}{\alpha_1+\beta_1}, \quad A_n = \frac{1}{\alpha_{n-1}+\beta_n} - \frac{1}{\alpha_n+\beta_n} \qquad (n = 2, 3, \ldots)$$

$$(16 \qquad B_n = \frac{1}{\alpha_n+\beta_n} - \frac{1}{\alpha_n+\beta_{n+1}} \qquad (n = 1, 2, \ldots).$$

Then the expression found for $f_1(x)$ takes the form

$$(17) \quad \left\{\begin{aligned} f_1(x) &= \sum_{n=1}^{\infty} \left[\frac{1}{\alpha_n+\beta_n} \{ U_n(x) - V_n(x) \} - \frac{1}{\alpha_n+\beta_{n+1}} \{ U_{n+1}(x) - V_n(x) \} \right] \\ &= -\sum_{n=1}^{\infty} A_n\, U_n(x) - \sum_{n=1}^{\infty} B_n\, V_n(x) \qquad (0 < x < \sigma). \end{aligned}\right.$$

Differentiating $U_n(x)$ and $V_n(x)$ with respect to x we find

$$(18) \qquad U_n'(x) = -\beta_n\, U_n(x)\ , \quad V_n'(x) = \alpha_n\, V_n(x).$$

Since the numbers A_n, B_n certainly are bounded and since the expressions $U_n(x)$, $V_n(x)$ are majorized by $e^{-(n-1)(\alpha_1+\beta_1)\sigma}$, it follows from (17) and (18) that $f_1(x)$ can be differentiated indefinitely for $0 < x < \sigma$.

On account of

$$e^{-\alpha_n\sigma-\beta_n\sigma-\beta_n x} < e^{-\alpha_n\sigma-\beta_n\sigma+\alpha_n x} < e^{-\beta_n x} \qquad (0 < x < \sigma)$$

the functions $U_n(x)$, $V_n(x)$ satisfy the relations

$$(19) \qquad U_{n+1}(x) < V_n(x) < U_n(x) \quad (n = 1, 2, \ldots).$$

We conclude this section by computing the following expressions

$$\sum_{k=1}^{n-1}(A_k + B_k),\quad \sum_{k=1}^{n-1}(-\beta_k A_k + \alpha_k B_k),\quad \sum_{k=1}^{n-1}(\beta_k^2 A_k + \alpha_k^2 B_k).$$

In order to shorten the calculations we introduce an unspecified positive number α_0 and for a moment replace

$$A_1 = -\frac{1}{\alpha_1+\beta_1} \quad \text{by} \quad \frac{1}{\alpha_0+\beta_1} - \frac{1}{\alpha_1+\beta_1}.$$

Then we find for $n = 1, 2, \ldots$

$$\sum_{k=1}^{n-1}(A_k + B_k) = \sum_{k=1}^{n-1}\left(\frac{1}{\alpha_{k-1}+\beta_k} - \frac{1}{\alpha_k+\beta_k} + \frac{1}{\alpha_k+\beta_k} - \frac{1}{\alpha_k+\beta_{k+1}}\right)$$
$$= \frac{1}{\alpha_0+\beta_1} - \frac{1}{\alpha_{n-1}+\beta_n},$$

$$\sum_{k=1}^{n-1}(-\beta_k A_k + \alpha_k B_k) = \sum_{k=1}^{n-1}\left(-\frac{\beta_k}{\alpha_{k-1}+\beta_k} + \frac{\beta_k}{\alpha_k+\beta_k} + \frac{\alpha_k}{\alpha_k+\beta_k} - \frac{\alpha_k}{\alpha_k+\beta_{k+1}}\right)$$
$$= \sum_{k=1}^{n-1}\left(-\frac{\beta_k}{\alpha_{k-1}+\beta_k} + 1 - \frac{\alpha_k}{\alpha_k+\beta_{k+1}}\right) = \sum_{k=1}^{n-1}\left(\frac{\alpha_{k-1}}{\alpha_{k-1}+\beta_k} - \frac{\alpha_k}{\alpha_k+\beta_{k+1}}\right)$$
$$= \frac{\alpha_0}{\alpha_0+\beta_1} - \frac{\alpha_{n-1}}{\alpha_{n-1}+\beta_n},$$

$$\sum_{k=1}^{n-1}(\beta_k^2 A_k + a_k^2 B_k) = \sum_{k=1}^{n-1}\left(\frac{\beta_k^2}{\alpha_{k-1}+\beta_k} - \frac{\beta_k^2}{\alpha_k+\beta_k} + \frac{\alpha_k^2}{\alpha_k+\beta_k} - \frac{\alpha_k^2}{\alpha_k+\beta_{k+1}}\right)$$
$$= \sum_{k=1}^{n-1}\left(\frac{\beta_k^2}{\alpha_{k-1}+\beta_k} - \beta_k + \alpha_k - \frac{\alpha_k^2}{\alpha_k+\beta_{k+1}}\right) = \sum_{k=1}^{n-1}\left(-\frac{\beta_k\alpha_{k-1}}{\alpha_{k-1}+\beta_k} + \frac{\alpha_k\beta_{k+1}}{\alpha_k+\beta_{k+1}}\right)$$
$$= -\frac{\alpha_0\beta_1}{\alpha_0+\beta_1} + \frac{\alpha_{n-1}\beta_n}{\alpha_{n-1}+\beta_n}.$$

Letting α_0 tend to infinity the final results take the form

$$\sum_{k=1}^{n-1}(A_k + B_k) = -\frac{1}{\alpha_{n-1}+\beta_n} \tag{20}$$

$$\sum_{k=1}^{n-1}(-\beta_k A_k + \alpha_k B_k) = 1 - \frac{\alpha_{n-1}}{\alpha_{n-1}+\beta_n} = \frac{\beta_n}{\alpha_{n-1}+\beta_n} \tag{21}$$

$$\sum_{k=1}^{n-1}(\beta_k^2 A_k + \alpha_k^2 B_k) = -\beta_1 + \frac{\alpha_{n-1}\beta_n}{\alpha_{n-1}+\beta_n} \qquad (n = 1, 2, \ldots). \tag{22}$$

4. *If $f(x)$ and $\varphi(x)$ fulfill the conditions* 3^0 *and* 4^0, *with some* $\sigma > 0$, *then* $f_1(x)$ *is logarithmic concave in the interior of the interval* $(0, \sigma)$

Since $f_1(x)$ certainly is twice differentiable for $0 < x < \sigma$, it comes to the same thing to show that

$$\frac{d^2}{dx^2}\log f_1(x) = \frac{f_1(x)f_1''(x) - \{f_1'(x)\}^2}{\{f_1(x)\}^2}$$

is at most equal to zero for $0<x<\sigma$. Put

$$H(x)=f_1(x)f_1''(x)-\{f_1'(x)\}^2.$$

Then, applying (17) and (18), for $0<x<\sigma$ we get

$$H(x)=\{\sum_{n=1}^{\infty}A_n U_n(x)+\sum_{n=1}^{\infty}B_n V_n(x)\}\cdot\{\sum_{n=1}^{\infty}\beta_n^2 A_n U_n(x)+\sum_{n=1}^{\infty}\alpha_n^2 B_n V_n(x)\}-$$
$$-\{-\sum_{n=1}^{\infty}\beta_n A_n U_n(x)+\sum_{n=1}^{\infty}\alpha_n B_n V_n(x)\}^2.$$

Carrying out the multiplications and gathering the terms with

$$U_n(x)\,U_k(x)\;,\;U_n(x)\,V_k(x)\;,\;V_n(x)\,V_k(x)\quad(n=1,2,\ldots;\;k=1,2,\ldots)$$

we find

$$(23)\qquad\begin{cases} H(x)=\sum\sum_{1\leqq k<n}(\beta_n-\beta_k)^2 A_n A_k U_n(x)\,U_k(x)\\ \quad+\sum\sum_{1\leqq k<n}(\beta_n+\alpha_k)^2 A_n B_k U_n(x)\,V_k(x)\\ \quad+\sum\sum_{1\leqq n\leqq k}(\beta_n+\alpha_k)^2 A_n B_k U_n(x)\,V_k(x)\\ \quad+\sum\sum_{1\leqq n\leqq k}(\alpha_k-\alpha_n)^2 B_n B_k V_n(x)\,V_k(x).\end{cases}$$

In virtue of (7), (15), (16) the numbers A_n, B_n are all $\geqq 0$, except A_1. Hence the first double series in the last member consists of non-negative terms only, since for each term $n\geqq 2$. Applying (19) we see, that the sum of this double series is not diminished, if in each term we replace $U_k(x)$ by $U_1(x)$. Similarly the sum of the second double series does not decrease if in each term we replace the factor $V_k(x)$ by $U_1(x)$. In the third double series the terms with $n=1$ may be negative, whereas the other terms are certainly non-negative; hence the sum of this series is not diminished if we replace $U_n(x)$ by $U_1(x)$. In the fourth double series we replace $V_n(x)$ by $U_1(x)$. We thus obtain

$$\frac{1}{U_1(x)}H(x)\leqq\sum\sum_{1\leqq k<n}\{(\beta_n-\beta_k)^2 A_k+(\beta_n+\alpha_k)^2 B_k\}A_n U_n(x)+$$
$$+\sum\sum_{1\leqq n\leqq k}\{(\beta_n+\alpha_k)^2 A_n+(\alpha_n-\alpha_k)^2 B_n\}B_k V_k(x).$$

Put

$$S_n^{(1)}=\sum_{k=1}^{n-1}\{(\beta_n-\beta_k)^2 A_k+(\beta_n+\alpha_k)^2 B_k\}\qquad(n=2,3,\ldots)$$

$$S_k^{(2)}=\sum_{n=1}^{k}\{(\beta_n+\alpha_k)^2 A_n+(\alpha_n-\alpha_k)^2 B_n\}\qquad(k=1,2,\ldots).$$

Then the last inequality takes the form

$$\frac{1}{U_1(x)}H(x)\leqq\sum_{n=2}^{\infty}A_n S_n^{(1)}U_n(x)+\sum_{k=1}^{\infty}B_k S_k^{(2)}V_k(x).$$

The expressions $S_n^{(1)}$ and $S_k^{(2)}$ can easily be computed. Applying (20), (21), (22) we find

$$S_n^{(1)} = \beta_n^2 \sum_{k=1}^{n-1} (A_k + B_k) + 2\beta_n \sum_{k=1}^{n-1} (-\beta_k A_k + \alpha_k B_k) + \sum_{k=1}^{n-1} (\beta_k^2 A_k + \alpha_k^2 B_k)$$

$$= -\frac{\beta_n^2}{\alpha_{n-1}+\beta_n} + \frac{2\beta_n^2}{\alpha_{n-1}+\beta_n} - \beta_1 + \frac{\alpha_{n-1}\beta_n}{\alpha_{n-1}+\beta_n} = -\beta_1 + \beta_n,$$

$$S_k^{(2)} = \alpha_k^2 \sum_{n=1}^{k} (A_n + B_n) - 2\alpha_k \sum_{n=1}^{k} (-\beta_n A_n + \alpha_n B_n) + \sum_{n=1}^{k} (\beta_n^2 A_n + \alpha_n^2 B_n)$$

$$= -\frac{\alpha_k^2}{\alpha_k+\beta_{k+1}} - \frac{2\alpha_k\beta_{k+1}}{\alpha_k+\beta_{k+1}} - \beta_1 + \frac{\alpha_k\beta_{k+1}}{\alpha_k+\beta_{k+1}} = -\beta_1 - \alpha_k.$$

Consequently

$$\frac{1}{U_1(x)} H(x) \leqq \sum_{n=2}^{\infty} (\beta_n - \beta_1) A_n U_n(x) + \sum_{k=1}^{\infty} (-\alpha_k - \beta_1) B_k V_k(x).$$

The first sum consists of non-negative terms only. In this sum put $n = k+1$. Using (19) we obtain

$$\frac{1}{U_1(x)} H(x) \leqq \sum_{k=1}^{\infty} (\beta_{k+1} - \beta_1) A_{k+1} U_{k+1}(x) + \sum_{k=1}^{\infty} (-\alpha_k - \beta_1) B_k V_k(x)$$

$$\leqq \sum_{k=1}^{\infty} \{(\beta_{k+1} - \beta_1) A_{k+1} + (-\alpha_k - \beta_1) B_k\} V_k(x).$$

Since

$$(\beta_{k+1} - \beta_1) A_{k+1} + (-\alpha_k - \beta_1) B_k$$

$$= \frac{\beta_{k+1}-\beta_1}{\alpha_k+\beta_{k+1}} - \frac{\beta_{k+1}-\beta_1}{\alpha_{k+1}+\beta_{k+1}} - \frac{\alpha_k+\beta_1}{\alpha_k+\beta_k} + \frac{\alpha_k+\beta_1}{\alpha_k+\beta_{k+1}}$$

$$= 1 - \frac{\beta_{k+1}-\beta_1}{\alpha_{k+1}+\beta_{k+1}} - \frac{\alpha_k+\beta_1}{\alpha_k+\beta_{k+1}} = \frac{\alpha_{k+1}+\beta_1}{\alpha_{k+1}+\beta_{k+1}} - \frac{\alpha_k+\beta_1}{\alpha_k+\beta_k},$$

we conclude

$$\frac{1}{U_1(x)} H(x) \leqq \sum_{k=1}^{\infty} t_k V_k(x) \qquad (0 < x < \sigma), \tag{24}$$

where

$$t_k = \frac{\alpha_{k+1}+\beta_1}{\alpha_{k+1}+\beta_{k+1}} - \frac{\alpha_k+\beta_1}{\alpha_k+\beta_k}. \tag{25}$$

The partial sums of these numbers t_k are all non-positive. In fact we have

$$\sum_{k=1}^{N} t_k = \frac{\alpha_{N+1}+\beta_1}{\alpha_{N+1}+\beta_{N+1}} - \frac{\alpha_1+\beta_1}{\alpha_1+\beta_1} = -\frac{\beta_{N+1}-\beta_1}{\alpha_{N+1}+\beta_{N+1}} \leqq 0 \qquad (N = 1, 2, \ldots).$$

Since for fixed x the values $V_k(x)$ form a decreasing sequence (see (19)), partial summation leads to

$$\sum_{k=1}^{\infty} t_k V_k(x) \leqq 0, \text{ hence } \frac{d^2}{dx^2} \log f_1(x) \leqq 0.$$

This proves the assertion.

MATHEMATICS

A PROPERTY OF LOGARITHMIC CONCAVE FUNCTIONS. II

BY

C. G. LEKKERKERKER

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of October 31, 1953)

5. *If $f(x)$ and $\varphi(x)$ satisfy the conditions* 3^0 *and* 4^0, *then $f_1(x)$ is logarithmic concave in the interval* $0 \leqq x < \infty$

Let g be a positive integer. Put

$$\varphi(g\sigma + x) = \varphi^*(x) \qquad (x \geqq 0).$$

Then $f(x)$ and $\varphi(x)$ fulfill the conditions 3^0 and 4^0, except for the relations (6). But the result of the preceding section remains true if (6) is not satisfied. Consequently the function $f_1^*(x)$, represented by

$$f_1^*(x) = \int_0^\infty f(t)\, \varphi^*(t+x)\, dt,$$

is logarithmic concave in the interval $0 < x < \sigma$. Since we have $f_1^*(x) = f_1(g\sigma + x)$ and since the positive number g is arbitrary, we conclude that $f_1(x)$ is logarithmic concave in the interior of each of the intervals $(g\sigma, (g+1)\sigma)$ $(g = 0, 1, 2, \ldots)$.

Now in section 2 we found that $f_1(x)$ is differentiable. Hence the last result means that $d/dx\{-\log f_1(x)\}$ is non-decreasing in each of the intervals $g\sigma < x < (g+1)\sigma$ $(g = 0, 1, 2, \ldots)$. Since $f_1(x)$ is even continuously differentiable for $x \geqq 0$, we conclude that $d/dx\{-\log f_1(x)\}$ is non-decreasing throughout for $x \geqq 0$. This proves that $f_1(x)$ is logarithmic concave in the interval $0 \leqq x < \infty$.

6. *The continuity and logarithmic concavity of $f_1(x)$ in the general case*

Let the functions $f(x)$ and $\varphi(x)$ be continuous for $x \geqq 0$ and fulfill the conditions 1^0 and 2^0. Suppose that (6) holds. We approximate $f(x)$ and $\varphi(x)$ by functions which possess the properties 3^0 and 4^0.

Let σ be an arbitrary positive number. We determine two functions $F(\sigma; x)$, $\Phi(\sigma; x)$ by the following requirements.

a) $F(\sigma; n\sigma) = f(n\sigma)$, $\Phi(\sigma; n\sigma) = \varphi(n\sigma)$ $(n = 0, 1, 2, \ldots)$

b) $\log F(\sigma; x)$ and $\log \Phi(\sigma; x)$ are linear functions of x in each of the intervals $(n-1)\sigma \leqq x \leqq n\sigma$ $(n = 1, 2, \ldots)$. Then the functions $F(\sigma; x)$, $\Phi(\sigma; x)$ are positive, continuous and steadily decreasing for $x \geqq 0$, whereas we have

$$F(\sigma; 0) = \Phi(\sigma; 0) = 1.$$

Put

$$(26)\quad \left\{\begin{array}{l} -\frac{1}{\sigma}[\log F(\sigma;\, n\sigma)-\log F(\sigma;\, (n-1)\sigma)]=\alpha_n \\ -\frac{1}{\sigma}[\log \Phi(\sigma;\, n\sigma)-\log \Phi(\sigma;\, (n-1)\sigma)]=\beta_n \end{array}\right\} \quad (n=1,\, 2,\, \ldots)$$

and define the numbers c_n, d_n $(n=1, 2, \ldots)$ by

$$F(\sigma;\, n\sigma)=f(n\sigma)=c_n\, e^{-\alpha_n n\sigma},$$
$$\Phi(\sigma;\, n\sigma)=\varphi(n\sigma)=d_n\, e^{-\beta_n n\sigma}.$$

This leads to the formulae

$$\left.\begin{array}{l} F(\sigma;\, x)=c_n\, e^{-\alpha_n x} \quad if \quad (n-1)\sigma \leqq x \leqq n\sigma \\ \Phi(\sigma;\, x)=d_n\, e^{-\beta_n x} \quad if \quad (n-1)\sigma \leqq x \leqq n\sigma \end{array}\right\} \quad (n=1,\, 2,\, \ldots).$$

We note that the numbers α_n, β_n, c_n, d_n depend on the choice of σ. Applying (1) to the function $g(x)=-\log f(x)$, with $x_1=(n-1)\sigma$, $x=n\sigma$, $x_2=(n+1)\sigma$, we find

$$g(n\sigma) \leqq \tfrac{1}{2}g((n-1)\sigma)+\tfrac{1}{2}g((n+1)\sigma),$$

.e.

$$-\log F(\sigma;\, n\sigma) \leqq -\tfrac{1}{2}\log F(\sigma;\, (n-1)\sigma)-\tfrac{1}{2}\log F(\sigma;\, (n+1)\sigma),$$

hence

$$\alpha_n \leqq \alpha_{n+1} \quad (n=1,\, 2,\, \ldots).$$

Since $F(\sigma; x)$ is steadily decreasing, the numbers α_n are positive. Similarly we can prove $0<\beta_1 \leqq \beta_2 \leqq \ldots$.

Consequently the functions $F(\sigma; x)$ and $\Phi(\sigma; x)$, determined by a) and b), possess the properties 3^0 and 4^0. Applying the results of the preceding sections we conclude that the function $F_1(\sigma; x)$, defined by

$$(27)\qquad F_1(\sigma;\, x)=\int_0^\infty F(\sigma;\, t)\, \Phi(\sigma;\, t+x)\,dt,$$

is logarithmic concave in the interval $0 \leqq x<\infty$.

We proceed to prove the relation

$$(28)\qquad \lim_{\sigma\to+0} F_1(\sigma;\, x)=\int_0^\infty f(t)\, \varphi(t+x)\,dt=f_1(x).$$

Let ε_1 be a positive number. Choose $\delta=\delta(\varepsilon_1)$, such that

$$\left.\begin{array}{l} |f(x')-f(x)| \\ |\varphi(x')-\varphi(x)| \end{array}\right\} <\varepsilon_1 \;\; if \;\; x-\delta<x'<x+\delta,\;\; x \geqq 0,\;\; x' \geqq 0.$$

Since for fixed $\tau>0$ the differences $f(t+\tau)-f(t)$, $\varphi(t+\tau)-\varphi(t)$ tend to zero as t tends to infinity, and since the functions $f(x)$ and $\varphi(x)$ are continuous and monotonic, this number δ can be chosen so as to be independent of x. Take $\sigma<\delta$ and let n be the positive integer with $(n-1)\sigma \leqq x<n\sigma$. Then we have

$$f((n-1)\sigma) \geqq f(x)>f(n\sigma) \quad , \quad \varphi((n-1)\sigma) \geqq \varphi(x)>\varphi(n\sigma)$$

and

$$f((n-1)\sigma) \geqq F(\sigma; x) > f(n\sigma) \ , \ \varphi((n-1)\sigma) \geqq \Phi(\sigma; x) > \varphi(n\sigma).$$

Hence

$$\left.\begin{array}{l} |F(\sigma; x) - f(x)| \\ |\Phi(\sigma; x) - \varphi(x)| \end{array}\right\} < \varepsilon_1 \ \textit{if} \ 0 < \sigma < \delta.$$

From a), b) and the logarithmic concavity of $f(x)$ and $\varphi(x)$ we deduce

$$F(\sigma; x) \leqq f(x) \ , \ \Phi(\sigma; x) \leqq \varphi(x) \ \textit{for} \ x \geqq 0.$$

Now let ε be a positive number. Choose $A > 0$, such that

$$\int_A^\infty f(t)\, \varphi(t+x)\, dt < \varepsilon \ \textit{for} \ x \geqq 0,$$

and take $\varepsilon_1 = (1/A)\varepsilon$. It follows that, for $\sigma < \delta = \delta((1/A)\varepsilon)$,

$$\begin{aligned} &|F_1(\sigma; x) - f_1(x)| = f_1(x) - F_1(\sigma; x) \\ &= \int_0^A \{f(t)\, \varphi(t+x) - F(\sigma; t)\, \Phi(\sigma; t+x)\} dt + \\ &+ \int_A^\infty \{f(t)\, \varphi(t+x) - F(\sigma; t)\, \Phi(\sigma; t+x)\} dt \\ &< f(0) \int_0^A \{\varphi(t+x) - \Phi(\sigma; t+x)\} dt + \Phi(\sigma; 0) \int_0^A \{f(t) - F(\sigma; t)\} dt + \\ &+ \int_A^\infty f(t)\, \varphi(t+x)\, dt \\ &< A\varepsilon_1 + A\varepsilon_1 + \varepsilon = 3\varepsilon. \end{aligned}$$

This proves the relation (28).

By what we have proved the inequality (1) holds, if for $g(x)$ we take the function $-\log F_1(\sigma; x)$. According to (28) the same inequality holds for each triple of real numbers x_1, x, x_2 with $0 \leqq x_1 < x < x_2$, if for $g(x)$ we take the function $f_1(x)$. Hence $f_1(x)$ is logarithmic concave in the interval $0 \leqq x < \infty$.

Since in the above analysis for each ε the numbers δ, A can be chosen so as to be independent of x, the relation (28) even holds uniformly for $x \geqq 0$. Now for each $\sigma > 0$ the function $F_1(\sigma; x)$ certainly is continuous. Hence $f_1(x)$ is continuous for $x \geqq 0$.

The proof of the main part of theorem 1 is now completed.

7. *Proof of the last part of theorem* 1

Let $f(x)$ and $\varphi(x)$ be continuous for $x \geqq 0$ and fulfill the conditions 1^0 and 2^0, whereas $f(0) = \varphi(0) = 1$. Put

$$-\log \varphi(x) = \psi(x) \ , \ -\log f_1(x) = g_1(x),$$

and suppose that $\psi(x)$ is not linear in any interval $a \leqq x < \infty$ $(a > 0)$. By a refinement of the foregoing proof we shall show that $g_1(x)$ even satisfies the condition

$$g_1(x) < \frac{x_2-x}{x_2-x_1}\, g_1(x_1) + \frac{x-x_1}{x_2-x_1}\, g_1(x_2) \;\; \textit{if} \;\; 0 \leqq x_1 < x < x_2.$$

Then the proof of theorem 1 will be completed.

Since $\psi(x)$ is convex, we have

$$\frac{\psi(x_2)-\psi(x_1)}{x_2-x_1} \leqq \frac{\psi(x_4)-\psi(x_3)}{x_4-x_3} \;\; \textit{if} \;\; 0 \leqq x_1 < x_2,\, x_3 < x_4.$$

Choose $a > 0$ arbitrary and put

$$\gamma = \inf_{a \leqq x_1 < x_2} \frac{\psi(x_2)-\psi(x_1)}{x_2-x_1}\,;$$

since $\psi(x)$ is steadily increasing, we have $\gamma > 0$. There further exist a positive number $b > a$ and a positive number $\gamma' > \gamma$, such that

$$\frac{\psi(x_2)-\psi(x_1)}{x_2-x_1} \geqq \gamma' \;\; \textit{if} \;\; b \leqq x_1 < x_2.$$

Let σ be a positive number $< \frac{1}{2}a$. We return to the functions $F(\sigma; x)$, $\Phi(\sigma; x)$, $F_1(\sigma; x)$, defined in the foregoing section, and the numbers α_n, β_n connected with these functions. Let $U_n(x)$, $V_n(x)$ $(n = 1, 2, \ldots)$ be defined by (13), (14), and put

$$H(x) = F_1(\sigma; x)\, F_1''(\sigma; x) - \{F_1'(\sigma; x)\}^2 \quad (x \neq 0, \sigma, 2\sigma, \ldots)$$

$$T_n = \sum_{k=1}^{n} t_k \quad (n = 1, 2, \ldots),$$

where t_k is defined by (25), so that

$$T_n = \frac{\alpha_{n+1}+\beta_1}{\alpha_{n+1}+\beta_{n+1}} - 1 = - \frac{\beta_{n+1}-\beta_1}{\alpha_{n+1}+\beta_{n+1}}.$$

To the expression $H(x)$ we can apply the result of section 4, viz. the relation (24). This gives

$$H(x) \leqq U_1(x) \sum_{k=1}^{\infty} t_k V_k(x) \;\; \textit{if} \;\; 0 < x < \sigma,$$

hence

$$H(x) \leqq U_1(x) \sum_{n=1}^{\infty} T_n \cdot \{V_n(x) - V_{n+1}(x)\}$$

$$= - U_1(x) \sum_{n=1}^{\infty} \frac{\beta_{n+1}-\beta_1}{\alpha_{n+1}+\beta_{n+1}} \cdot \{V_n(x) - V_{n+1}(x)\} \;\; \textit{if} \;\; 0 < x < \sigma.$$

In order to obtain an estimate for $H(x)$ in the intervals $g\sigma < x < (g+1)\sigma$, where g is a non-negative integer, we consider the function

$$\Phi^*(\sigma; x) = \frac{1}{\Phi(\sigma; g\sigma)}\, \Phi(\sigma; x - g\sigma).$$

We have

$$\Phi^*(\sigma; x) = d_n^* e^{-\beta_n^* x} \;\; if \;\; (n-1)\sigma \leqq x \leqq n\sigma \;\; (n=1, 2, \dots),$$

where d_n^*, β_n^* are positive constants with

$$d_1^* = \Phi^*(\sigma; 0) = 1 \;\; , \;\; \Phi^*(\sigma; n\sigma) = d_n^* e^{-\beta_n^* n\sigma},$$
$$\beta_n^* = \beta_{n+g} \quad (n = 1, 2, \dots).$$

Starting with the pair of functions $F(\sigma; x)$, $\Phi^*(\sigma; x)$ in stead of the pair $F(\sigma; x)$, $\Phi(\sigma; x)$ we can form the corresponding expression $H(x)$. To this expression we can apply the estimate found above. Stating the result in terms of the original functions we find that in the interval $g\sigma < x < (g+1)\sigma$ the expression $H(x)$ satisfies the estimate

$$(29) \qquad H(x) \leqq - U_1^{(g)}(x) \sum_{n=1}^{\infty} \frac{\beta_{n+g+1} - \beta_{g+1}}{\alpha_{n+1} + \beta_{n+g+1}} \cdot \{V_n^{(g)}(x) - V_{n+1}^{(g)}(x)\},$$

where

$$U_1^{(g)}(x) = \Phi(\sigma; g\sigma)\, e^{-\beta_1^* x} = e^{-(\beta_1 + \dots + \beta_g)x - \beta_{g+1}x},$$
$$V_n^{(g)}(x) = \Phi(\sigma; g\sigma) \cdot e^{-(\alpha_1 + \dots + \alpha_n)\sigma - (\beta_1^* + \dots + \beta_g^*)\sigma + \alpha_n x}$$
$$= e^{-(\alpha_1 + \dots + \alpha_n)\sigma - (\beta_1 + \dots + \beta_{n+g})\sigma + \alpha_n x}.$$

Put

$$K = \left[\frac{a}{\sigma}\right] - 1, \; N = \left[\frac{b}{\sigma}\right] + 1$$

and let g run through the integers 0, 1, ..., K. Then we have $(K+1)\sigma \leqq a$, hence

$$\beta_n = \frac{\psi(n\sigma) - \psi((n-1)\sigma)}{\sigma} \leqq \gamma \;\; for \;\; n = 1, 2, \dots, K+1.$$

Consequently

$$U_1^{(g)}(x) > e^{-(\beta_1 + \dots + \beta_g + \beta_{g+1})\sigma}$$
$$\geqq e^{-\gamma(g+1)\sigma} \geqq e^{-\gamma a} \;\; for \;\; g = 0, 1, \dots, K.$$

For $n \geqq N$ we have $(n+g)\sigma \geqq N\sigma > b$, hence

$$\beta_{n+g+1} = \frac{\psi((n+g+1)\sigma) - \psi((n+g)\sigma)}{\sigma} \geqq \gamma'.$$

Henceforth

$$\beta_{n+g+1} - \beta_{g+1} \geqq \gamma' - \gamma \;\; for \;\; n = N, N+1, \dots; \; g = 0, 1, \dots, K.$$

Now let n in particular have one of the values $N, N+1, \dots, 2N-1$, so that $(n+g+1)\sigma \leqq (2N+K)\sigma \leqq 2b + 2\sigma + a < 2b + 2a$.

There exists a fixed number δ', only depending on a, b and the functions $f(x)$, $\varphi(x)$, but independent of σ, such that

$$\left.\begin{matrix}\alpha_{n+1} \\ \beta_{n+g+1}\end{matrix}\right\} \leqq \delta' \;\; for \;\; n = N, N+1, \dots, 2N-1; \; g = 0, 1, \dots, K.$$

Finally for $V_N^{(g)}(x) - V_{2N}^{(g)}(x)$ we get the estimate

$$V_N^{(g)}(x) - V_{2N}^{(g)}(x) = e^{-(\alpha_1+\ldots+\alpha_N)\sigma-(\beta_1+\ldots+\beta_{N+g})\sigma+\alpha_N x}$$
$$\cdot\{1 - e^{-(\alpha_{N+1}+\ldots+\alpha_{2N})\sigma-(\beta_{N+g+1}+\ldots+\beta_{2N+g})\sigma+\alpha_{2N}x-\alpha_N x}\}$$
$$> e^{-(2N+g)\delta'\sigma}\cdot\{1 - e^{-(\beta_{N+g+1}+\ldots+\beta_{2N+g})\sigma}\}$$
$$\geqq e^{-(2N+K)\delta'\sigma}\cdot\{1 - e^{-N\gamma'\sigma}\}$$
$$> e^{-(2b+2a)\delta'}\cdot(1 - e^{-b\gamma'})$$

or $g=0, 1, \ldots, K$ and $g\sigma < x < (g+1)\sigma$.

Using all these estimates we deduce from (29)

$$-H(x) \geqq U_1^{(g)}(x) \sum_{n=1}^{\infty} \frac{\beta_{n+g+1}-\beta_{g+1}}{\alpha_{n+1}+\beta_{n+g+1}} \cdot \{V_n^{(g)}(x) - V_{n+1}^{(g)}(x)\}$$

$$> e^{-\gamma a}\sum_{n=N}^{2N-1} \frac{\beta_{n+g+1}-\beta_{g+1}}{\alpha_{n+1}+\beta_{n+g+1}} \cdot \{V_n^{(g)}(x) - V_{n+1}^{(g)}(x)\}$$

$$> e^{-\gamma a} \cdot \frac{\gamma'-\gamma}{2\delta'} \sum_{n=N}^{2N-1} \{V_n^{(g)}(x) - V_{n+1}^{(g)}(x)\}$$

$$= e^{-\gamma a} \cdot \frac{\gamma'-\gamma}{2\delta'} \{V_N^{(g)}(x) - V_{2N}^{(g)}(x)\},$$

hence

$$-H(x) \geqq \frac{\gamma'-\gamma}{2\delta'} e^{-\gamma a-(2b+2a)\delta'}\cdot(1 - e^{-b\gamma'}), \tag{30}$$

if $g\sigma < x < (g+1)\sigma$ and g is one of the integers $0, 1, \ldots, K$.

The right hand member of (30) does not depend on σ. Noting the definition of $H(x)$, we conclude that for each positive number a there exists a positive number A, independent of σ, such that

$$\frac{d^2}{dx^2}\{-\log F_1(\sigma; x)\}$$

exists and is at least equal to A for each pair of real numbers σ and x with

$$0 < \sigma < \tfrac{1}{2}a,\ 0 < x < \tfrac{1}{2}a\ ,\ \frac{x}{\sigma} \text{ not integral.}$$

Writing $-\log F_1(\sigma; x) = g_1(\sigma; x)$ it follows that

$$\frac{1}{x_3-x_2}\left\{\frac{g_1(\sigma; x_4)-g_1(\sigma; x_3)}{x_4-x_3} - \frac{g_1(\sigma; x_2)-g_1(\sigma; x_1)}{x_2-x_1}\right\} \geqq A,$$

if x_1, x_2, x_3, x_4 are positive numbers with $0 < x_1 < x_2 < x_3 < x_4 < \frac{1}{2}a$ and if $0 < \sigma < \frac{1}{2}a$. Applying the relation (28) we see that the last inequality remains true if we replace $g_1(\sigma; x)$ by $g_1(x) = -\log f_1(x)$. Hence $g_1(x)$ is strictly convex in the interval $(0, \frac{1}{2}a)$. Since a was arbitrary, this proves that $f_1(x)$ is strictly logarithmic concave in the interval $0 \leqq x < \infty$.

This completes the proof of theorem 1.

Proof of theorem 2. The number γ is finite (see note [2])). Applying theorem 1 with $\varphi(x) = f(x)$ we see that

$$\frac{2}{\gamma}\int_0^{\infty} f(t)\, f(t+x)\, dt$$

exists for $x \geqq 0$ and, as a function of x, is positive, continuous, steadily decreasing and logarithmic concave in the interval $0 \leqq x < \infty$. Consequently the relations (4) define inductively a sequence of functions $f_n(x)$ which are all positive, continuous, steadily decreasing and logarithmic concave in the interval $0 \leqq x < \infty$.

By hypothesis $\log f(x)$ is not a linear function of x in the interval $0 \leqq x < \infty$. Then there also exists a positive number a, such that $\log f(x)$ is not linear in the interval $a \leqq x < \infty$. Inspecting the proof of the last part of theorem 1 we find that $\log f_1(x)$ is not a linear function of x in the interval $(0, \frac{1}{2}a)$. A fortiori $\log f_1(x)$ is not a linear function of x in the interval $0 \leqq x < \infty$. Hence all functions $f_n(x)$ have the property that $\log f_n(x)$ is not a linear function of x in the interval $0 \leqq x < \infty$.

Let n be a non-negative integer. There exists a positive number α, such that $f_n(x) = O(e^{-\alpha x})$ as $x \to \infty$. So we may deduce

$$\int_0^\infty f_{n+1}(x)\, dx = \frac{2}{\gamma} \int_0^\infty \int_0^\infty f_n(t)\, f_n(t+x)\, dt\, dx$$

$$= \frac{2}{\gamma} \int_0^\infty \int_0^\infty f_n(t)\, f_n(t+x)\, dx\, dt = \frac{2}{\gamma} \iint_{u \geqq t \geqq 0} f_n(t)\, f_n(u)\, du\, dt,$$

hence

$$\int_0^\infty f_{n+1}(x)\, dx = \frac{1}{\gamma} \iint_{\substack{u \geqq 0 \\ t \geqq 0}} f_n(t)\, f_n(u)\, du\, dt = \frac{1}{\gamma} \{ \int_0^\infty f_n(u)\, du \}^2.$$

Hence on account of (3) we find

$$\int_0^\infty f_n(x) dx = \gamma \text{ for } n = 0, 1, 2, \ldots.$$

Consequently, in order to complete the proof of theorem 2, it is sufficient to prove the relation

$$\frac{2}{\gamma} \int_0^\infty \{f(t)\}^2 dt > f(0),$$

or, stated otherwise,

$$2 \int_0^\infty \{f(t)\}^2 dt - f(0) \int_0^\infty f(t) dt > 0. \tag{31}$$

The left hand member of (31) is homogeneous (of degree 2) in f. Consequently we may suppose without loss of generality $f(0) = 1$.

As in the proof of theorem 1 let σ be a positive number and let the function $F(\sigma; x)$ be defined by

a) $F(\sigma; n\sigma) = f(n\sigma) \quad (n = 0, 1, 2, \ldots)$

b) $\log F(\sigma; x)$ is linear in each interval $(n-1)\sigma \leqq x \leqq n\sigma$ $(n = 1, 2, \ldots)$.

Then there exist positive constants c_n, α_n $(n = 1, 2, \ldots)$, such that

$$F(\sigma; x) = c_n\, e^{-\alpha_n x} \text{ if } (n-1)\, \sigma \leqq x \leqq n\sigma$$
$$0 < \alpha_1 \leqq \alpha_2 \leqq \ldots$$
$$c_n\, e^{-\alpha_n n\sigma} = c_{n+1}\, e^{-\alpha_{n+1} n\sigma} = F(\sigma; n\sigma)$$
$$c_1 = F(\sigma; 0) = 1.$$

Hence we can deduce

$$2\int_0^\infty \{F(\sigma;x)\}^2\,dx - F(\sigma;0)\int_0^\infty F(\sigma;x)\,dx$$

$$= 2\sum_{n=1}^{\infty} \frac{c_n}{2\alpha_n}\{e^{-2\alpha_n(n-1)\sigma} - e^{-2\alpha_n n\sigma}\} - \sum_{n=1}^{\infty}\frac{c_n}{\alpha_n}\{e^{-\alpha_n(n-1)\sigma} - e^{-\alpha_n n\sigma}\}$$

$$= \sum_{n=1}^{\infty}\frac{1}{\alpha_n}[F^2(\sigma;(n-1)\sigma) - F^2(\sigma;n\sigma) - F(\sigma;(n-1)\sigma) + F(\sigma;n\sigma)]$$

$$= \frac{1}{\alpha_1}\{F^2(\sigma;0) - F(\sigma;0)\} - \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right)\cdot\{F^2(\sigma;n\sigma) - F(\sigma;n\sigma)\}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right)\cdot\{F(\sigma;n\sigma) - F^2(\sigma;n\sigma)\}.$$

On account of $\alpha_n \leqq \alpha_{n+1}$ the last expression certainly is non-negative. But we can say more. Since $\log f(x)$ is not linear in the interval $0 \leqq x < \infty$, there exist positive numbers a, b, γ, γ', such that

$$2a < b\,,\ \gamma < \gamma'$$

$$\alpha_n \leqq \gamma \quad if \quad n \leqq \left[\frac{a}{\sigma}\right]$$

$$\alpha_n \geqq \gamma' \quad if \quad n \geqq \left[\frac{b}{\sigma}\right] + 1.$$

If σ has any value with $0 < \sigma < \frac{1}{2}a$ and n is a positive integer with $[(a/\sigma)] \leqq n \leqq [(b/\sigma)]$, then

$$F(\sigma;n\sigma) - F^2(\sigma;n\sigma) = F(\sigma;n\sigma)\cdot\{1 - F(\sigma;n\sigma)\}$$

$$\geqq F\left(\sigma;\left[\frac{b}{\sigma}\right]\sigma\right)\cdot\left\{1 - F\left(\sigma;\left[\frac{a}{\sigma}\right]\sigma\right)\right\} > f(b)\cdot\{1 - f(\tfrac{1}{2}a)\}.$$

Hence

$$2\int_0^\infty \{F(\sigma;x)\}^2\,dx - F(\sigma;0)\int_0^\infty F(\sigma;x)\,dx$$

$$\geqq \sum_{n=[\sigma^{-1}a]}^{n=[\sigma^{-1}b]}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right)\cdot\{F(\sigma;n\sigma) - F^2(\sigma;n\sigma)\}$$

$$> f(b)\cdot\{1 - f(\tfrac{1}{2}a)\}\sum_{n=[\sigma^{-1}a]}^{n=[\sigma^{-1}b]}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right)$$

$$\geqq f(b)\cdot\{1 - f(\tfrac{1}{2}a)\}\cdot\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right).$$

The last expression is positive and does not depend on σ. We now apply the relation (28); this gives

$$2\int_0^\infty \{f(x)\}^2\,dx - f(0)\int_0^\infty f(x)\,dx \geqq f(b)\cdot\{1 - f(\tfrac{1}{2}a)\}\cdot\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right).$$

This proves (31) and so completes the proof of theorem 2.

Mathematisch Centrum, Amsterdam